

# A Grothendieck-Witt space for stable infinity categories with duality

Markus Spitzweck

November 1, 2016

## Abstract

We construct a Grothendieck-Witt space for any stable infinity category with duality. If we apply our construction to perfect complexes over a commutative ring in which 2 is invertible we recover the classical Grothendieck-Witt space. Our Grothendieck-Witt space is a grouplike  $E_\infty$ -space which is part of a genuine  $C_2$ -spectrum, the connective real  $K$ -theory spectrum.

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Recollections and preliminaries</b>	<b>2</b>
<b>3</b>	<b>The Grothendieck-Witt space</b>	<b>3</b>
<b>4</b>	<b>The comparison</b>	<b>5</b>
<b>5</b>	<b>The zeroth Grothendieck-Witt group</b>	<b>10</b>
<b>6</b>	<b>Hyperbolic categories</b>	<b>14</b>
<b>7</b>	<b>A connective real <math>K</math>-theory spectrum</b>	<b>15</b>

## 1 Introduction

In this paper we carry over the hermitian  $S_\bullet$ -construction which can be found e.g. in [15] to the  $\infty$ -categorical setting. The input of our construction is an  $\infty$ -category with duality in the sense of [10] whose underlying  $\infty$ -category is stable. For such an  $\infty$ -category  $C$  we define a Grothendieck-Witt space  $\mathrm{GW}(C)$  which has the structure of a grouplike  $E_\infty$ -space, so gives rise to a connective spectrum. We show in the last section that this spectrum is in fact part of a genuine  $C_2$ -spectrum  $\mathrm{KR}(C)$ , the connective real  $K$ -theory spectrum of  $C$ .

This way we obtain for example for any  $E_\infty$ -ring spectrum  $R$  and tensor invertible object  $L \in \mathrm{Perf}(R)$  (here  $\mathrm{Perf}(R)$  denotes the stable  $\infty$ -category of perfect  $R$ -modules) a real  $K$ -theory spectrum  $\mathrm{KR}(R, L)$ , in particular spectra

$\mathrm{KR}(R, R[n])$  for any  $n \in \mathbb{Z}$ , by considering the  $L$ -twisted duality on  $\mathrm{Perf}(R)$  (see [10, §8]).

To justify our constructions we prove in section 4 that our Grothendieck-Witt space is equivalent to the classical Grothendieck-Witt space (as defined e.g. in [15]) in the case that  $R$  is a discrete ring in which 2 is invertible and  $L$  a shifted invertible (in the discrete sense)  $R$ -module.

**Acknowledgements:** I would like to thank Hongyi Chu, David Gepner, Hadrian Heine, Kristian Moi, Thomas Nikolaus, Oliver Röndigs, Manfred Stelzer, Sean Tilson and Girja Tripathi for very helpful discussions and suggestions on the subject.

## 2 Recollections and preliminaries

We use the same conventions as in [10]. In particular  $\mathrm{Cat}_\infty^{hC_2}$  is the  $\infty$ -category of small  $\infty$ -categories with duality. We denote by  $\mathrm{Cat}_\infty^{\mathrm{st}}$  the subcategory of  $\mathrm{Cat}_\infty$  of stable  $\infty$ -categories and exact functors between them.

It follows from [10, Proposition 2.2] that the induced functor

$$(\mathrm{Cat}_\infty^{\mathrm{st}})^{hC_2} \rightarrow \mathrm{Cat}_\infty^{hC_2}$$

(on the left we use the induced  $C_2$ -action) is a monomorphism in  $\widehat{\mathrm{Cat}}_\infty$  whose good image consists of those  $\infty$ -categories with duality whose underlying  $\infty$ -category is stable and those functors between  $\infty$ -categories with duality whose underlying functor is exact. We write for this good image  $(\mathrm{Cat}_\infty^{hC_2})^{\mathrm{st}}$ .

Usually we will not distinguish between a category and its nerve viewed as an  $\infty$ -category.

We will frequently see objects  $[n] \in \Delta$  as categories. For a category  $C$  we write  $\mathrm{Ar}(C)$  for the arrow category, i.e. the functor category  $\mathrm{Fun}([1], C)$ .

**Proposition 2.1.** *Let  $n \in \mathbb{N}$  and  $C_1, \dots, C_n$  be stable  $\infty$ -categories. Let  $D$  be an  $\infty$ -category which admits finite limits and denote by  $\mathrm{Sp}(D)$  the stabilization of  $D$ . Let*

$$\mathrm{Fun}'(C_1 \times \dots \times C_n, D) \subset \mathrm{Fun}(C_1 \times \dots \times C_n, D)$$

*be the full subcategory on those functors which preserve finite limits separately in each variable, and let*

$$\mathrm{Fun}'(C_1 \times \dots \times C_n, \mathrm{Sp}(D)) \subset \mathrm{Fun}(C_1 \times \dots \times C_n, \mathrm{Sp}(D))$$

*be the full subcategory on those functors which are exact separately in each variable. Then composition with the functor  $\Omega^\infty: \mathrm{Sp}(D) \rightarrow D$  induces an equivalence*

$$\mathrm{Fun}'(C_1 \times \dots \times C_n, \mathrm{Sp}(D)) \rightarrow \mathrm{Fun}'(C_1 \times \dots \times C_n, D)$$

*of  $\infty$ -categories. If  $C_1 = \dots = C_n$  then this equivalence respects the  $\Sigma_n$ -actions.*

*Proof.* This follows from [13, Corollary 1.4.2.23.].  $\square$

Let  $C \in \mathrm{Cat}_\infty$ . To give a duality on  $C$  (or equivalently on  $C^{\mathrm{op}}$ ) is the same as to give a  $C_2$ -homotopy fixed point of  $\mathrm{Fun}(C \times C, \mathrm{Spc})$  (or equivalently of

$\text{Fun}(C^{\text{op}} \times C^{\text{op}}, \mathbf{Spc})$  which is underlying a perfect pairing (see [10, Corollary 7.3]). If  $\varphi \in \text{Fun}(C^{\text{op}} \times C^{\text{op}}, \mathbf{Spc})$  corresponds to a duality then  $\varphi$  is informally given by  $(X, Y) \mapsto \text{map}(X, Y^\vee)$ .

If  $C$  is now stable it follows from Proposition 2.1 that such a  $C_2$ -homotopy fixed point is the same as a  $C_2$ -homotopy fixed point  $B$  of  $\text{Fun}(C^{\text{op}} \times C^{\text{op}}, \mathbf{Sp})$  which is nondegenerate representable in the sense of [11]. The bilinear functor  $B$  is informally given by  $(X, Y) \mapsto \text{map}^{\text{Sp}}(X, Y^\vee)$ , where  $\text{map}^{\text{Sp}}$  denotes the mapping spectrum functor.

### 3 The Grothendieck-Witt space

Let  $C \in \mathbf{Cat}_\infty^{\text{st}}$ . Building on Waldhausen's definition we define for any  $n \in \mathbb{N}$  the  $\infty$ -category  $S_n(C)$  to be the full subcategory of the functor category  $\text{Fun}(\text{Ar}([n]), C)$  on those functors  $A$  such that for any  $0 \leq i \leq n$  the object  $A_{i,i}$  is a zero object in  $C$  and such that for any  $0 \leq i \leq j \leq k \leq n$  the square

$$\begin{array}{ccc} A_{i,j} & \longrightarrow & A_{i,k} \\ \downarrow & & \downarrow \\ A_{j,j} & \longrightarrow & A_{j,k} \end{array}$$

is exact in  $C$ . These properties are preserved by the suspension and loop functors on  $C$ , thus the  $\infty$ -categories  $S_n(C)$  are stable. The simplicial  $\infty$ -category  $\text{Fun}(\text{Ar}([\bullet]), C)$  restricts to a simplicial  $\infty$ -category  $S_\bullet(C)$ . Taking levelwise core groupoids yields the simplicial object  $S_\bullet^\sim(C)$  in spaces whose realization we denote by  $|S_\bullet^\sim(C)|$ .

**Definition 3.1.** *The  $K$ -theory space  $K(C)$  of the stable  $\infty$ -category  $C$  is defined to be the loop space  $\Omega|S_\bullet^\sim(C)|$ , where we take a zero object of  $C$  as base point.*

**Remark 3.2.** *The space  $K(C)$  has the natural structure of a grouplike  $E_\infty$ -space (see also the discussion at the end of this section for the case of the Grothendieck-Witt space).*

Our  $\infty$ -categorical definition of the Grothendieck-Witt space of a stable  $\infty$ -category with duality is modelled on the hermitian  $S_\bullet$ -construction given for example in [15]. This uses the edgewise subdivision of a simplicial object which we introduce now.

**Definition 3.3.** *Let  $X: \Delta^{\text{op}} \rightarrow C$ ,  $[n] \mapsto X_n$ , be a simplicial object in an  $\infty$ -category  $C$ . Then the edgewise subdivision  $E(X)$  is defined to be the simplicial object  $X \circ \iota^{\text{op}}$ , where  $\iota: \Delta \rightarrow \Delta$  is the endofunctor defined by  $[n] \mapsto [n]^{\text{op}} * [n]$ .*

Thus we have  $E(X)_n = X_{2n+1}$ . The inclusions  $[n] \hookrightarrow [n]^{\text{op}} * [n]$  define a natural transformation from the identity functor on  $\Delta$  to  $\iota$  and thus we are at the disposal of a natural map of simplicial objects  $E(X) \rightarrow X$ .

For  $C$  a stable  $\infty$ -category we let  $S_\bullet^e(C) := E(S_\bullet(C))$ , and likewise  $S_\bullet^{e,\sim}(C) := E(S_\bullet^\sim(C))$ .

For each  $n \in \mathbb{N}$  the category  $[n]$  has a unique structure of a category with strict duality and the assignment  $[n] \mapsto [n]^{\text{op}} * [n]$  can be viewed as a functor

from  $\Delta$  to the category of categories with strict duality  $\mathbf{CD}$ , therefore the same holds for the assignment  $[n] \mapsto \mathbf{Ar}([n]^{\text{op}} * [n])$ .

In [10, §11] a functor  $\epsilon: \mathbf{CD} \rightarrow \mathbf{Cat}_{\infty}^{hC_2}$  is constructed. Moreover  $\mathbf{Cat}_{\infty}^{hC_2}$  is cartesian closed, and the internal hom commutes with the forgetful functor  $\mathbf{Cat}_{\infty}^{hC_2} \rightarrow \mathbf{Cat}_{\infty}$ .

Thus for an  $\infty$ -category  $C$  with duality  $\mathbf{Fun}(\mathbf{Ar}([n]^{\text{op}} * [n]), C)$  is an object of  $\mathbf{Cat}_{\infty}^{hC_2}$  functorial in  $[n]$ .

If now  $C$  is a stable  $\infty$ -category with duality then for any  $n \in \mathbb{N}$  the full subcategory  $S_n(C)$  of  $\mathbf{Fun}(\mathbf{Ar}([n]), C)$  is preserved by the duality (since the dual of an exact square is again an exact square), thus  $S_{\bullet}^e(C)$  can be viewed as a simplicial object in  $\mathbf{Cat}_{\infty}^{hC_2}$ , and  $S_{\bullet}^{e,\sim}(C)$  can be viewed as a simplicial object in  $\mathbf{Spc}^{hC_2} \simeq \mathbf{Spc}[C_2]$ .

Taking levelwise the homotopy  $C_2$ -fixed points of the latter object defines the simplicial space  $(S_{\bullet}^{e,\sim}(C))_h$ .

**Definition 3.4.** For  $C$  a stable  $\infty$ -category with duality we let the Grothendieck-Witt space  $\mathbf{GW}(C)$  of  $C$  be the homotopy fiber of the composition

$$|(S_{\bullet}^{e,\sim}(C))_h| \rightarrow |S_{\bullet}^{e,\sim}(C)| \rightarrow |S_{\bullet}^{\sim}(C)|.$$

We now equip  $\mathbf{GW}(C)$  with an  $E_{\infty}$ -structure which will turn out to be grouplike (i.e. an infinite loop space structure), see Proposition 5.8. The monomorphisms

$$\mathbf{Cat}_{\infty}^{\text{st}} \rightarrow \mathbf{Cat}_{\infty}^{\text{preadd}} \rightarrow \mathbf{Cat}_{\infty}$$

as well as the full embedding

$$\mathbf{Cat}_{\infty}^{\text{preadd}} \rightarrow \mathbf{SymMonCat}_{\infty}$$

of  $\infty$ -categories carry  $C_2$ -actions (see [10, §6]), hence we have an induced composition

$$s: (\mathbf{Cat}_{\infty}^{hC_2})^{\text{st}} \simeq (\mathbf{Cat}_{\infty}^{\text{st}})^{hC_2} \rightarrow (\mathbf{Cat}_{\infty}^{\text{preadd}})^{hC_2} \rightarrow \mathbf{SymMonCat}_{\infty}^{hC_2}.$$

Thus, since  $S_{\bullet}^e(C)$  is in fact naturally a simplicial object in  $(\mathbf{Cat}_{\infty}^{hC_2})^{\text{st}}$ , we obtain a simplicial object  $s(S_{\bullet}^e(C))$  in  $\mathbf{SymMonCat}_{\infty}^{hC_2}$ , and applying the functor

$$\mathbf{SymMonCat}_{\infty}^{hC_2} \xrightarrow{(-)^{\sim}} \mathbf{Mon}_{E_{\infty}}(\mathbf{Spc})[C_2]$$

yields a lift of  $S_{\bullet}^{e,\sim}(C)$  to a simplicial object of  $\mathbf{Mon}_{E_{\infty}}(\mathbf{Spc})[C_2]$ .

Similarly the map

$$S_{\bullet}^{e,\sim}(C) \rightarrow S_{\bullet}^{\sim}(C)$$

lifts to a map between simplicial objects in  $\mathbf{Mon}_{E_{\infty}}(\mathbf{Spc})$ .

Denoting the lifts with the same symbols we obtain maps

$$(S_{\bullet}^{e,\sim}(C))_h \rightarrow S_{\bullet}^{e,\sim}(C) \rightarrow S_{\bullet}^{\sim}(C)$$

of simplicial objects in  $\mathbf{Mon}_{E_{\infty}}(\mathbf{Spc})$ . Taking realizations and the fiber of the induced composition equips  $\mathbf{GW}(C)$  with a natural  $E_{\infty}$ -structure.

## 4 The comparison

We denote the (hermitian)  $S_\bullet$ -construction used in [15] for an exact category with weak equivalences (and duality)  $\mathcal{E}$  by the same symbols as we used in the  $\infty$ -categorical situation except that we write  $S^{\text{str}}$  instead of  $S$ . Thus for example if  $\mathcal{E}$  has a duality then  $S_\bullet^{\text{str},e}(\mathcal{E})$  is a simplicial exact category with weak equivalences and duality and  $S_\bullet^{\text{str},e,\sim}(\mathcal{E})$  denotes the simplicial subcategory of weak equivalences.

We denote by

$$|\cdot|: \text{Cat}^1 \rightarrow \text{Spc}$$

the natural functor from the 1-category of small categories  $\text{Cat}^1$  to the  $\infty$ -category of spaces which takes the realization of the nerve. Thus if  $C_\bullet$  is for example a simplicial category then  $|C_\bullet|$  will be a simplicial object in  $\text{Spc}$ . The realization of this simplicial object is denoted by  $|C_\bullet|_r$ .

The Grothendieck-Witt space  $\text{GW}(\mathcal{E})$  is then defined to be the homotopy fiber of the natural map

$$|(S_\bullet^{\text{str},e,\sim}(\mathcal{E}))_h|_r \rightarrow |S_\bullet^{\text{str},\sim}(\mathcal{E})|_r.$$

As in the  $\infty$ -categorical case we can equip  $\text{GW}(\mathcal{E})$  with a natural  $E_\infty$ -structure (use that the functor  $\mathcal{H}^{\text{lax}}$  (see [10, §11]) is symmetric monoidal for the cartesian symmetric monoidal structures since it is a right adjoint).

For a ring  $R$  we denote by  $\mathcal{P}_R$  the category of finitely generated projective  $R$ -modules and by  $\text{Cpx}^b(\mathcal{P}_R)$  the exact category with weak equivalences of bounded complexes with values in  $\mathcal{P}_R$ . We denote by  $\text{Perf}(R)$  the stable  $\infty$ -category of perfect  $R$ -modules.

Note that we exhibit a natural functor  $\text{Cpx}^b(\mathcal{P}_R) \rightarrow \text{Perf}(R)$  which is a localization at the quasi isomorphisms.

We now assume that  $R$  is commutative, fix for the whole section an integer  $N \in \mathbb{Z}$  and an invertible  $R$ -module  $L$  and equip  $\text{Cpx}^b(\mathcal{P}_R)$  with the strong duality  $X \mapsto \underline{\text{Hom}}(X, L[N])$ .

[10, Corollary 8.5] equips  $\text{Perf}(R)$  with the duality given by the object  $L[N] \in \text{Pic}(\text{Perf}(R))$ , and the naturality of the construction of loc. cit. shows that the functor

$$\text{Cpx}^b(\mathcal{P}_R) \rightarrow \text{Perf}(R)$$

preserves the dualities.

To emphasize the dependence on the duality we denote the corresponding Grothendieck-Witt spaces by  $\text{GW}(\text{Cpx}^b(\mathcal{P}_R), N, L)$  and  $\text{GW}(\text{Perf}(R), N, L)$ .

For any  $n \in \mathbb{N}$  we obtain a functor

$$\text{Fun}(\text{Ar}([n]), \text{Cpx}^b(\mathcal{P}_R)) \rightarrow \text{Fun}(\text{Ar}([n]), \text{Perf}(R))$$

between  $\infty$ -categories with duality.

The restriction to the full subcategory  $S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R))$  of this functor factors through  $S_n(\text{Perf}(R))$  yielding functors

$$\begin{aligned} S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R)) &\rightarrow S_n(\text{Perf}(R)) \\ S_n^{\text{str},\sim}(\text{Cpx}^b(\mathcal{P}_R)) &\rightarrow S_n^{\sim}(\text{Perf}(R)) \end{aligned} \tag{1}$$

between  $\infty$ -categories with duality.

The functors

$$\mathcal{H}^{\text{lax}}(S_n^{\text{str}, \sim}(\text{Cpx}^b(\mathcal{P}_R))) \rightarrow \mathcal{H}^{\text{lax}}(S_n^{\sim}(\text{Perf}(R)))$$

induced on lax hermitian objects (see [10, §11]) by the latter functors together with the equivalence

$$\mathcal{H}(S_n^{\sim}(\text{Perf}(R))) \simeq \mathcal{H}^{\text{lax}}(S_n^{\sim}(\text{Perf}(R)))$$

yields functors

$$(S_n^{\text{str}, \sim}(\text{Cpx}^b(\mathcal{P}_R)))_h \rightarrow (S_n^{\sim}(\text{Perf}(R)))_h \quad (2)$$

(see also [10, Proposition 11.8]).

Every map in  $S_n^{\text{str}, \sim}(\text{Cpx}^b(\mathcal{P}_R))$  is sent to an equivalence under the functor (1), thus we obtain maps

$$|S_n^{\text{str}, \sim}(\text{Cpx}^b(\mathcal{P}_R))| \rightarrow S_n^{\sim}(\text{Perf}(R)) \quad (3)$$

in  $\text{Spc}[C_2]$ . Also every map in  $(S_n^{\text{str}, \sim}(\text{Cpx}^b(\mathcal{P}_R)))_h$  is sent to an equivalence under the functor (2), thus we obtain maps

$$|(S_n^{\text{str}, \sim}(\text{Cpx}^b(\mathcal{P}_R)))_h| \rightarrow (S_n^{\sim}(\text{Perf}(R)))_h \quad (4)$$

in  $\text{Spc}$ .

After edgewise subdivision we get a map

$$|(S_{\bullet}^{\text{str}, e, \sim}(\text{Cpx}^b(\mathcal{P}_R)))_h| \rightarrow (S_{\bullet}^{e, \sim}(\text{Perf}(R)))_h$$

between simplicial objects in  $\text{Spc}$ .

Altogether we arrive at a commutative square

$$\begin{array}{ccc} |(S_{\bullet}^{\text{str}, e, \sim}(\text{Cpx}^b(\mathcal{P}_R)))_h| & \longrightarrow & |S_{\bullet}^{\text{str}, \sim}(\text{Cpx}^b(\mathcal{P}_R))| \\ \downarrow & & \downarrow \\ (S_{\bullet}^{e, \sim}(\text{Perf}(R)))_h & \longrightarrow & S_{\bullet}^{\sim}(\text{Perf}(R)) \end{array}$$

of simplicial objects in  $\text{Spc}$ , which yields after taking realizations and fibers of the horizontal induced maps the comparison map

$$\text{GW}(\text{Cpx}^b(\mathcal{P}_R), N, L) \rightarrow \text{GW}(\text{Perf}(R), N, L). \quad (5)$$

**Remark 4.1.** *The comparison map (5) can be made compatible with the  $E_{\infty}$ -structures on both sides. We leave the details to the interested reader.*

**Theorem 4.2.** *If 2 is invertible in  $R$  the comparison map (5) is an equivalence.*

*Proof.* Combine the next two Lemmas. □

**Lemma 4.3.** *The maps (3) are equivalences.*

*Proof.* This is standard. □

The main input to our comparison statement is

**Lemma 4.4.** *If 2 is invertible in  $R$  then the maps (4) are equivalences.*

*Proof.* We have a commutative diagram

$$\begin{array}{ccc}
|(S_n^{\text{str}, \sim}(\text{Cpx}^b(\mathcal{P}_R)))_h| & \longrightarrow & S_n^{\sim}(\text{Perf}(R))_h \\
\downarrow & & \downarrow \\
|S_n^{\text{str}, \sim}(\text{Cpx}^b(\mathcal{P}_R))| & \longrightarrow & S_n^{\sim}(\text{Perf}(R))
\end{array} \tag{6}$$

in  $\mathbf{Spc}$ . We want to show that the upper horizontal map is an equivalence. By Lemma 4.3 the lower horizontal map is an equivalence. We will show that for any  $X \in S_n^{\text{str}, \sim}(\text{Cpx}^b(\mathcal{P}_R))$  the space of paths  $\text{map}(X, X^\vee)$  in  $|S_n^{\text{str}, \sim}(\text{Cpx}^b(\mathcal{P}_R))|$  (or equivalently in  $S_n^{\sim}(\text{Perf}(R))$ ) carries a natural  $C_2$ -action, that the homotopy fibers of the vertical maps in the diagram over  $X$  (resp. the image of  $X$ ) are canonically identified with  $\text{map}(X, X^\vee)^{hC_2}$  and that the induced map (by the commutative square) on these fibers respect these identifications. From this the claim follows.

We first apply [15, Lemma 4] to the exact category  $S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R))$  with duality (considering only the isomorphisms as weak equivalences) to obtain a category  $C = S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R))_{\text{iso}}^{\text{str}}$  with a strict duality which is equivalent to  $S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R))$  as category with (strong) duality. We denote by  $C^\sim$  the subcategory of  $C$  of weak equivalences (which correspond to the objectwise quasi isomorphisms in  $S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R))$ ).

For a category  $D$  we let  $\text{Tw}(D)$  be the twisted arrow category of  $D$  whose objects are the morphisms of  $D$ , and a map from  $f: A \rightarrow B$  to  $g: C \rightarrow D$  is a commutative square

$$\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow f & & \downarrow g \\
B & \longleftarrow & D
\end{array}$$

in  $D$ . If  $D$  has a strict duality then the assignment  $f \mapsto f^\vee$  defines a (strict)  $C_2$ -action on  $\text{Tw}(D)$  whose (strict)  $C_2$ -fixed points is the category of hermitian objects of  $D$ .

Similarly for an  $\infty$ -category  $D$  the assignment

$$\Delta^{\text{op}} \ni [n] \mapsto \text{map}([n] \star [n]^{\text{op}}, D)$$

defines a complete Segal space whose associated  $\infty$ -category is defined to be the twisted arrow category  $\text{Tw}(D)$  of  $D$  (this is compatible with the 1-categorical definition). If  $D$  has a duality then the above assignment has values in  $\mathbf{Spc}[C_2]$ , thus  $\text{Tw}(D)$  has a  $C_2$ -action. Moreover by the construction in [10, §11] we have a canonical equivalence

$$\text{Tw}(D)^{hC_2} \simeq \mathcal{H}^{\text{lax}}(D).$$

The canonical map  $\text{Tw}(D) \rightarrow D \times D^{\text{op}}$  is  $C_2$ -equivariant, where the action on  $D \times D^{\text{op}}$  is given by  $(X, Y) \mapsto (Y^\vee, X^\vee)$ , and we have  $(D \times D^{\text{op}})^{hC_2} \simeq D$ .

The right vertical map of diagram (6) can thus be identified with the map

$$\text{Tw}(S_n^{\sim}(\text{Perf}(R)))^{hC_2} \rightarrow (S_n^{\sim}(\text{Perf}(R)) \times S_n^{\sim}(\text{Perf}(R))^{\text{op}})^{hC_2}.$$

We have a commutative diagram

$$\begin{array}{ccc} \mathrm{Tw}(C^\sim) & \longrightarrow & \mathrm{Tw}(S_n^\sim(\mathrm{Perf}(R))) \\ \downarrow & & \downarrow \\ C^\sim \times (C^\sim)^{\mathrm{op}} & \longrightarrow & S_n^\sim(\mathrm{Perf}(R)) \times S_n^\sim(\mathrm{Perf}(R))^{\mathrm{op}} \end{array}$$

in  $\mathrm{Cat}_\infty[C_2]$ . In the induced diagram

$$\begin{array}{ccc} |\mathrm{Tw}(C^\sim)| & \longrightarrow & \mathrm{Tw}(S_n^\sim(\mathrm{Perf}(R))) \\ \downarrow & & \downarrow \\ |C^\sim \times (C^\sim)^{\mathrm{op}}| & \longrightarrow & S_n^\sim(\mathrm{Perf}(R)) \times S_n^\sim(\mathrm{Perf}(R))^{\mathrm{op}} \end{array}$$

in  $\mathrm{Spc}[C_2]$  the horizontal maps are equivalences. Thus diagram (6) can be identified with the diagram

$$\begin{array}{ccc} |\mathrm{Tw}(C^\sim)^{C_2}| & \longrightarrow & |\mathrm{Tw}(C^\sim)|^{hC_2} \\ \downarrow & & \downarrow \\ |(C^\sim \times (C^\sim)^{\mathrm{op}})^{C_2}| & \longrightarrow & |C^\sim \times (C^\sim)^{\mathrm{op}}|^{hC_2} \end{array} \quad (7)$$

whose lower entries can be identified with  $|C^\sim|$ .

For  $X \in C$  we let  $P_X$  be defined by the (strict) pullback diagram

$$\begin{array}{ccc} P_X & \longrightarrow & \mathrm{Tw}(C^\sim) \\ \downarrow & & \downarrow \\ (C^\sim \times (C^\sim)^{\mathrm{op}})/(X, X^\vee) & \longrightarrow & C^\sim \times (C^\sim)^{\mathrm{op}} \end{array} \quad (8)$$

of categories. Since  $(X, X^\vee) \in C^\sim \times (C^\sim)^{\mathrm{op}}$  is a fixed point with respect to the  $C_2$ -action  $P_X$  inherits a  $C_2$ -action and this diagram becomes  $C_2$ -equivariant. Taking  $C_2$ -fixed points of this diagram gives a diagram canonically isomorphic to the pullback diagram

$$\begin{array}{ccc} P_X^{C_2} & \longrightarrow & C_h^\sim \\ \downarrow & & \downarrow \\ C^\sim/X & \longrightarrow & C^\sim. \end{array} \quad (9)$$

For a map  $X \rightarrow Y$  in  $C^\sim$  we have an induced  $C_2$ -equivariant map  $P_X \rightarrow P_Y$ .

*Claim 1:* For any map  $X \rightarrow Y$  in  $C^\sim$  the map  $|P_X| \rightarrow |P_Y|$  is an equivalence.

*Claim 2:* For any  $X \in C$  the map  $|P_X^{C_2}| \rightarrow |P_X|^{hC_2}$  is an equivalence.

*Claim 3:* For any map  $X \rightarrow Y$  in  $C^\sim$  the map  $|P_X^{C_2}| \rightarrow |P_Y^{C_2}|$  is an equivalence.

Claim 3 follows from Claims 1 and 2.

It follows from Claim 1 and Quillen's Theorem B (dual of [8, Theorem 5.6]) that the realization of diagram (8) is a pullback diagram, similarly it follows



from Claim 3 and Quillen's Theorem B that the realization of diagram (9) is a pullback diagram.

Thus the induced map on the homotopy fibers over an  $X \in C$  of the vertical maps in diagram (7) can be identified with the map  $|P_X^{C_2}| \rightarrow |P_X|^{hC_2}$  (for the second fiber note that homotopy fixed points preserve fiber sequences) which is an equivalence by Claim 2. So we see that if we prove Claims 1 and 2 the proof is finished.

*Proof of Claim 1:* It follows from [4, Propositions 6.2 and 8.2] and the correction [3] that  $|P_X|$  is canonically equivalent to the mapping space  $\text{map}(X, X^\vee)$  in  $|C^\sim|$ . The category  $P_X$  is naturally isomorphic to the category denoted  $C^\sim(X, X^\vee)_{\text{Hom-tw}}$  in [3], and this a collection of connected components of  $C(X, X^\vee)_{\text{Hom-tw}}$ . The factorizations necessary for these arguments are given by cylinder constructions. Thus the claim follows.

*Proof of Claim 2:* Let  $X \in C$  and  $X'$  be the image of  $X$  in  $S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R))$ . Let  $N\mathbb{Z}[\Delta^\bullet]$  be the cosimplicial object in  $\text{Cpx}^b(\mathcal{P}_\mathbb{Z})$  which assigns to  $[n]$  the complex corresponding to the simplicial abelian group  $\mathbb{Z}[\Delta^\bullet]$  under the Dold-Kan correspondence. Thus it is Reedy cofibrant, and the cosimplicial object  $N\mathbb{Z}[\Delta^\bullet] \otimes X'$  in  $S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R))$  is a special cosimplicial resolution of  $X'$  in the sense of [5]. We let  $X^\bullet$  be the image of this cosimplicial object in  $C$ . Also let  $(C^\sim/X)_f$  be the full subcategory of  $C^\sim/X$  on those maps  $Y \rightarrow X$  which are surjections. Then by [5, Proposition 6.12] the functor  $\varphi: \Delta \rightarrow (C^\sim/X)_f$  which sends  $[n]$  to  $X^n \twoheadrightarrow X$  is left cofinal.

Let the categories  $R$  and  $S$  be defined by the pullback diagram

$$\begin{array}{ccccc} S & \xrightarrow{\psi} & R & \xrightarrow{i} & P_X^{C_2} \\ \downarrow & & \downarrow & & \downarrow \\ \Delta & \xrightarrow{\varphi} & (C^\sim/X)_f & \longrightarrow & C^\sim/X. \end{array} \quad (10)$$

Let  $r \in R$ , so  $r$  consists of a surjection  $Y \twoheadrightarrow X$  in  $C^\sim$  together with a hermitian structure on  $Y$ . Since for any map  $Z \rightarrow Y$  in  $C^\sim$  there exists a unique hermitian structure on  $Z$  compatible with the one on  $Y$  it follows that the natural functor

$$\psi/r \rightarrow \varphi/(Y \twoheadrightarrow X)$$

is an isomorphism, hence  $\psi$  is also left cofinal.

The vertical functors in diagram (10) are right fibrations, and the fiber over an object  $Y \rightarrow X$  is  $\text{Hom}_{C^\sim}(Y, Y^\vee)^{C_2}$  (i.e. the set of hermitian structures on  $Y$ ). Thus for a vertical map  $A \rightarrow B$  in this diagram we exhibit a functor  $j_B: B^{\text{op}} \rightarrow \text{Set} \hookrightarrow \mathbf{sSet}$ , and  $|A|$  is naturally equivalent to  $\text{hocolim} j_B$ . It follows that  $|S| \rightarrow |R|$  is an equivalence.

Let  $K$  be the simplicial set defined by  $[n] \mapsto \text{Hom}_C(X^n, (X^n)^\vee)$  ( $K$  has then in fact the structure of a simplicial  $R$ -module) and  $K^\sim$  the subsimplicial set on those simplices which are in  $C^\sim$ . The simplicial set  $K$  has a natural  $C_2$ -action and  $K^\sim$  is stable under this action.

It follows from the above considerations that  $|S|$  is naturally equivalent to  $(K^\sim)^{C_2}$ .

Note that the natural map

$$K^{C_2} \rightarrow K^{hC_2}$$

is an weak homotopy equivalence since 2 is invertible in  $R$ , thus, since  $K^\sim$  consists of certain connected components of  $K$ , the same follows for the map

$$(K^\sim)^{C_2} \rightarrow (K^\sim)^{hC_2}.$$

For a map  $f: Y \rightarrow Z$  in  $S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R))$  denote by  $c(f)$  the construction [16, 1.5.5] applied to the map  $f^{\text{op}}$  in  $S_n^{\text{str}}(\text{Cpx}^b(\mathcal{P}_R))^{\text{op}}$ . We therefore obtain a factorization  $Y \rightarrow c(f) \rightarrow Z$  of  $f$  into an inclusion which is a quasi isomorphism followed by a surjection, and moreover there is a retraction  $c(f) \rightarrow Y$  of the first map. This construction is functorial in  $f$ . If  $Y$  has a hermitian structure then the retraction induces a hermitian structure on  $c(f)$ . These constructions can be transported to  $C$ .

For an object  $a \in P_X^{C_2}$  with underlying object  $f: Y \rightarrow X$  in  $C^\sim/X$  the above factorization applied to  $f$  yields an object  $(c(f) \rightarrow X) \in (C^\sim/X)_f$  and also an object in  $R$  (using the induced hermitian structure on  $c(f)$ ). This assignment defines a functor  $p: P_X^{C_2} \rightarrow R$  together with natural transformations  $\text{id} \rightarrow i \circ p$  and  $\text{id} \rightarrow p \circ i$ . It follows that the realization of  $i$  is an equivalence.

Hence we have seen that the natural map  $|S| \rightarrow |P_X^{C_2}|$  in  $\mathbf{Spc}$  is an equivalence, and that in the commutative square

$$\begin{array}{ccc} (K^\sim)^{C_2} & \longrightarrow & |P_X^{C_2}| \\ \downarrow & & \downarrow \\ (K^\sim)^{hC_2} & \longrightarrow & |P_X|^{hC_2} \end{array}$$

in  $\mathbf{Spc}$  the upper horizontal and the left vertical maps are equivalences. Also the lower horizontal map is an equivalence. Hence Claim 2 and thus the Lemma are proved.  $\square$

## 5 The zeroth Grothendieck-Witt group

Let  $C \in \text{Cat}_\infty^{hC_2}$ . Then the right fibration

$$p: \text{Tw}(C) \rightarrow C \times C^{\text{op}}$$

inherits a  $C_2$ -action (see the proof of Lemma 4.4). Thus for  $X \in C$  the space  $\text{map}(X, X^\vee)$  has a natural  $C_2$ -action, since it arises as the homotopy fiber of  $p$  over a homotopy fixed point for the  $C_2$ -action.

Using the equivalence  $\text{Tw}(C)^{hC_2} \simeq \mathcal{H}^{\text{lax}}(C)$  we see that the fiber over  $X$  of the functor  $\mathcal{H}^{\text{lax}}(C) \rightarrow C$  is the  $\infty$ -groupoid  $\text{map}(X, X^\vee)^{hC_2}$ , so to give a lax hermitian structure on  $X$  is the same as to give a  $C_2$ -homotopy fixed point of  $\text{map}(X, X^\vee)$ .

On the other hand the symmetric functor  $C^{\text{op}} \times C^{\text{op}} \rightarrow \mathbf{Spc}$  corresponding to the duality on  $C$  can be viewed as a map in  $\widehat{\text{Cat}}_\infty[C_2]$  (where the  $C_2$ -action on the source is the switch map and on the target the trivial action), and taking homotopy fixed points yields a functor

$$C^{\text{op}} \rightarrow \mathbf{Spc}[C_2]$$

which is informally given by  $X \mapsto \text{map}(X, X^\vee)$ . This way  $\text{map}(X, X^\vee)$  also inherits a  $C_2$ -action which can be seen to be naturally equivalent to the action from above.

If  $C$  is stable the same argument as above yields a functor

$$C^{\text{op}} \rightarrow \text{Sp}[C_2]$$

which is informally given by  $X \mapsto \text{map}^{\text{Sp}}(X, X^\vee)$ . Composing with

$$\Omega^\infty: \text{Sp} \rightarrow \text{Spc}$$

yields the functor above. Let  $Q: C^{\text{op}} \rightarrow \text{Sp}$  be given by  $X \mapsto \text{map}^{\text{Sp}}(X, X^\vee)^{hC_2}$ . We see that a lax hermitian structure on an  $X \in C$  is the same as a quadratic object structure of  $(C, Q)$  on  $X$  in the sense of [12], and a hermitian structure on  $X$  is same as a Poincare object structure on  $X$ .

**Lemma 5.1.** *Let  $I \in \text{Cat}_\infty$  and  $C \in \text{Cat}_\infty^{hC_2}$ . Then there is a natural functor*

$$\mathcal{H}(\text{Fun}(\mathcal{G}^{\text{lax}}(I), C)) \rightarrow \text{Fun}(I, \mathcal{H}^{\text{lax}}(C))$$

*inducing an equivalence on core groupoids.*

*Proof.* Functorially in  $J \in \text{Cat}_\infty$  we have a chain of maps

$$\begin{aligned} \text{map}(J, \mathcal{H}(\text{Fun}(\mathcal{G}^{\text{lax}}(I), C))) &\simeq \text{map}(\mathcal{G}(J), \text{Fun}(\mathcal{G}^{\text{lax}}(I), C)) \\ &\simeq \text{map}(\mathcal{G}(J) \times \mathcal{G}^{\text{lax}}(I), C) \rightarrow \text{map}(\mathcal{G}^{\text{lax}}(J) \times \mathcal{G}^{\text{lax}}(I), C) \\ &\rightarrow \text{map}(\mathcal{G}^{\text{lax}}(J \times I), C) \simeq \text{map}(J \times I, \mathcal{H}^{\text{lax}}(C)) \simeq \text{map}(J, \text{Fun}(I, \mathcal{H}^{\text{lax}}(C))) \end{aligned}$$

in  $\text{Spc}$  defining the functor in question. Taking core groupoids the functor reduces to the equivalence

$$\text{map}(\mathcal{G}^{\text{lax}}(I), C) \simeq \text{map}(I, \mathcal{H}^{\text{lax}}(C)).$$

□

**Remark 5.2.** *In general the functor in Lemma 5.1 is not an equivalence, since in general the map  $\mathcal{G}^{\text{lax}}(J \times I) \rightarrow \mathcal{G}(J) \times \mathcal{G}^{\text{lax}}(I)$  is not an equivalence. We always have an equivalence*

$$\mathcal{H}(\text{Fun}(\mathcal{G}(I), C)) \simeq \text{Fun}(I, \mathcal{H}(C))$$

*of  $\infty$ -categories.*

Let  $C \in \text{Cat}_\infty^{hC_2}$  and  $\alpha \in \mathcal{H}(\text{Fun}([1], C))$  (so  $\alpha$  can be identified with a lax hermitian object of  $C$ ). Let  $F_\alpha$  be fiber over  $\alpha$  of the functor

$$\mathcal{H}(\text{Fun}([3], C)) \rightarrow \mathcal{H}(\text{Fun}([1], C))$$

induced by the duality preserving functor  $[1] \rightarrow [3]$  which sends 0 to 1 and 1 to 2.

Because of Lemma 5.1 the core groupoid  $F_\alpha^\sim$  can then be identified with  $(\mathcal{H}^{\text{lax}}(C)_{/\alpha})^\sim$  (use  $\mathcal{G}^{\text{lax}}([0]) \simeq [1]$  and  $\mathcal{G}^{\text{lax}}([1]) \simeq [3]$ ). Since  $\mathcal{H}^{\text{lax}}(C) \rightarrow C$  is a right fibration the latter category can be identified with  $(C_{/\alpha(0)})^\sim$ .

The duality preserving functor  $[3] \rightarrow [2]$  given by  $0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 2$  exhibits  $\text{Fun}([2], C)$  as the full subcategory of  $\text{Fun}([3], C)$  of those functors for

which the middle induced map is an equivalence. Therefore we also get a full embedding

$$\mathcal{H}(\text{Fun}([2], C)) \rightarrow \mathcal{H}(\text{Fun}([3], C)).$$

The considerations above show that the core groupoid of the fiber of the functor

$$\mathcal{H}(\text{Fun}([2], C)) \rightarrow \mathcal{H}(C)$$

induced by  $[0] \rightarrow [2]$ ,  $0 \mapsto 1$ , over a hermitian object  $X$  is naturally equivalent to  $(C_{/X})^\sim$  (here  $X$  also denotes the object underlying the hermitian object).

**Proposition 5.3.** *Let  $C \in (\text{Cat}_\infty^{hC_2})^{\text{st}}$  and  $X \in \mathcal{H}(C)$ . Let  $F$  be the fiber over  $X$  of the functor*

$$\mathcal{H}(S_2(C)) \rightarrow \mathcal{H}(C)$$

*induced by the inclusion  $[0] \rightarrow \text{Ar}([2])$ ,  $0 \mapsto \text{id}_2$ . Then the fiber of the natural map*

$$F^\sim \rightarrow (C_{/X})^\sim$$

*over a map  $f: Y \rightarrow X$  is naturally equivalent to the subspace of the space of paths in  $\text{map}_C(Y, Y^\vee)^{hC_2}$  from the zero map to the map*

$$Y \xrightarrow{f} X \simeq X^\vee \xrightarrow{f^\vee} Y^\vee$$

*on those connected components which exhibit the resulting commutative square*

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Y^\vee \end{array}$$

*as an exact square in  $C$ .*

*Proof.* This follows from the above considerations together with the next two Lemmas.  $\square$

**Lemma 5.4.** *Let  $D$  be the category associated to the 1-skeleton of  $[1] \times [1]$  (so  $D$  is Joyal-equivalent to two  $\Lambda_1^2$ 's glued together along their endpoints) and  $i: D \rightarrow [1] \times [1]$  the natural map. Then for an  $\infty$ -category  $C$  the fiber of the functor*

$$i^*: \text{Fun}([1] \times [1], C) \rightarrow \text{Fun}(D, C)$$

*over an object  $\alpha: D \rightarrow C$  in the target is naturally equivalent to the space of paths in  $\text{map}(X, Y)$  ( $X$  being  $\alpha((0, 0))$  and  $Y$  being  $\alpha((1, 1))$ ) from the composition of one composable pair of maps in  $D$  to the composition of the other composable pair.*

*Proof.* This follows from the fact that there is a pushout square

$$\begin{array}{ccc} E & \longrightarrow & D \\ \downarrow & & \downarrow \\ [1] & \longrightarrow & [1] \times [1], \end{array}$$

where  $E$  is obtained by gluing two copies of  $[1]$  together along their endpoints.  $\square$

**Lemma 5.5.** *Let  $C$  be an  $\infty$ -category which has a zero object. Let  $i: [2] \rightarrow [1] \times [1]$  be the map which sends 0 to  $(0,0)$ , 1 to  $(0,1)$  and 2 to  $(1,1)$ . Let*

$$\mathrm{Fun}'([1] \times [1], C) \subset \mathrm{Fun}([1] \times [1], C)$$

*be the full subcategory on those squares such that the entry in spot  $(1,0)$  is a zero object. Then the fiber of the functor*

$$\mathrm{Fun}'([1] \times [1], C) \rightarrow \mathrm{Fun}([2], C)$$

*over an object  $\alpha: [2] \rightarrow C$  is naturally equivalent to the space of paths in the mapping space  $\mathrm{map}(\alpha(0), \alpha(2))$  from the zero map to  $\alpha(0 \rightarrow 2)$ .*

*If  $C$  has a duality and  $\alpha \in \mathcal{H}(\mathrm{Fun}([2], C))$ , then  $\mathrm{map}(\alpha(0), \alpha(2))$  has a natural  $C_2$ -action, the map  $\alpha(0 \rightarrow 2)$  naturally lies in  $\mathrm{map}(\alpha(0), \alpha(2))^{hC_2}$  and the fiber of the functor*

$$\mathcal{H}(\mathrm{Fun}'([1] \times [1], C)) \rightarrow \mathcal{H}(\mathrm{Fun}([2], C))$$

*is naturally equivalent to the space of paths in  $\mathrm{map}(\alpha(0), \alpha(2))^{hC_2}$  from the zero map to  $\alpha(0 \rightarrow 2)$ .*

*Proof.* The first part follows from Lemma 5.4, the second by taking hermitian objects.  $\square$

**Corollary 5.6.** *Let  $C \in (\mathrm{Cat}_\infty^{hC_2})^{\mathrm{st}}$  and  $X \in \mathcal{H}(C)$ . Let  $F$  be the fiber over  $X$  of the functor*

$$\mathcal{H}(S_2(C)) \rightarrow \mathcal{H}(C)$$

*induced by the inclusion  $[0] \rightarrow \mathrm{Ar}([2])$ ,  $0 \mapsto \mathrm{id}_2$ . Then giving a point in  $F$  is the same as giving a Lagrangian of  $X$  in the sense of [12, Example 7.].*

**Corollary 5.7.** *Let  $C \in (\mathrm{Cat}_\infty^{hC_2})^{\mathrm{st}}$ ,  $X \in \mathcal{H}(C)$  and  $\varphi: X \rightarrow X^\vee$  the corresponding map in*

$$\mathrm{map}(X, X^\vee)^{hC_2} \simeq \Omega^\infty \mathrm{map}^{\mathrm{Sp}}(X, X^\vee)^{hC_2}.$$

*Choose an inverse  $-\varphi$  of  $\varphi$  with respect to the infinite loop space structure on  $\mathrm{map}(X, X^\vee)^{hC_2}$ . Then there is an object of  $\mathcal{H}(S_2(C))$  whose underlying exact triangle in  $C$  has the form*

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{diag}} & X \oplus X \\ \downarrow & & \downarrow \varphi + (-\varphi) \\ 0 & \longrightarrow & X^\vee \end{array}$$

*and where the hermitian structure on  $X \oplus X$  is given by  $\varphi \oplus (-\varphi)$ .*

*Proof.* This follows now from the proof of [12, Proposition 11.].  $\square$

**Proposition 5.8.** *For  $C \in (\mathrm{Cat}_\infty^{hC_2})^{\mathrm{st}}$  the  $E_\infty$ -structure on  $\mathrm{GW}(C)$  defined in section 3 is grouplike.*

*Proof.* There is a coequalizer diagram

$$\pi_0(S_1^{e,\sim}(C)_h) \rightrightarrows \pi_0(S_0^{e,\sim}(C)_h) \rightarrow \pi_0(|S_{\bullet}^{e,\sim}(C)_h|)$$

in **Set**. Let a point in  $\pi_0(S_0^{e,\sim}(C)_h)$  be represented by a hermitian object  $X \in \mathcal{H}(C)$ . Let  $\varphi \in \text{map}(X, X^\vee)^{hC_2}$  be the corresponding map with a choice of an inverse  $-\varphi$ . Let  $W' \in \mathcal{H}(S_2(C))$  be the object described in Corollary 5.7 and  $W$  be the image of  $W'$  under the functor  $\mathcal{H}(S_2(C)) \rightarrow \mathcal{H}(S_3(C))$  induced by the map  $[3] \rightarrow [2]$  given by  $0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 2$ . The object  $W$  determines an element of  $\pi_0(S_1^{e,\sim}(C)_h)$  which is sent under the two maps above to 0 resp.  $(X \oplus X, \varphi \oplus (-\varphi))$ . So we see that in  $\pi_0(|S_{\bullet}^{e,\sim}(C)_h|)$  an inverse of the image of  $(X, \varphi)$  is given by  $(X, -\varphi)$ , in particular the  $E_\infty$ -space  $|S_{\bullet}^{e,\sim}(C)_h|$  is grouplike. It follows that also  $\text{GW}(C)$  is grouplike.  $\square$

**Definition 5.9.** For  $n \in \mathbb{N}$  the abelian group  $\text{GW}_n(C) := \pi_n \text{GW}(C)$  is called the  $n$ -th Grothendieck-Witt group of the stable  $\infty$ -category  $C$  with duality. In particular the group  $\text{GW}_0(C)$  is called the Grothendieck-Witt group of  $C$ .

## 6 Hyperbolic categories

We denote a right adjoint of the forgetful functor

$$\text{Cat}_\infty^{hC_2} \rightarrow \text{Cat}_\infty$$

by  $\text{Hyp}$  and call  $\text{Hyp}(C)$  the hyperbolic category associated to  $C \in \text{Cat}_\infty$ . The underlying category of  $\text{Hyp}(C)$  is equivalent to  $C \times C^{\text{op}}$ , and the duality is informally given by  $C \times C^{\text{op}} \ni (X, Y) \mapsto (Y, X)$ .

**Lemma 6.1.** Let  $C \in \text{Cat}_\infty$ . Then there is a natural equivalence

$$\mathcal{H}^{\text{lax}}(\text{Hyp}(C)) \simeq \text{Tw}(C)$$

of  $\infty$ -categories.

*Proof.* The  $\infty$ -category  $\mathcal{H}^{\text{lax}}(\text{Hyp}(C))$  is given as the complete Segal space

$$[n] \mapsto \text{map}_{\text{Cat}_\infty^{hC_2}}([n] * [n]^{\text{op}}, \text{Hyp}(C)),$$

which by adjunction is equivalent to

$$[n] \mapsto \text{map}_{\text{Cat}_\infty}([n] * [n]^{\text{op}}, C).$$

But this is a possible definition of the twisted arrow category  $\text{Tw}(C)$ .  $\square$

**Corollary 6.2.** There is a natural equivalence

$$\mathcal{H}(\text{Hyp}(C))^\sim \simeq C^\sim$$

in  $\text{Spc}$  for  $C \in \text{Cat}_\infty$ .

*Proof.* The core groupoid of the full subcategory of  $\text{Tw}(C)$  on the equivalences is naturally equivalent to  $C^\sim$ .  $\square$

**Lemma 6.3.** *There is a natural equivalence*

$$\mathrm{Fun}(I, \mathrm{Hyp}(C)) \simeq \mathrm{Hyp}(\mathrm{Fun}(I, C))$$

in  $\mathrm{Cat}_\infty^{hC_2}$  for  $I \in \mathrm{Cat}_\infty^{hC_2}$  and  $C \in \mathrm{Cat}_\infty$ .

*Proof.* By adjunction a map

$$\mathrm{Fun}(I, \mathrm{Hyp}(C)) \rightarrow \mathrm{Hyp}(\mathrm{Fun}(I, C))$$

is the same as a map

$$\mathrm{Fun}(I, \mathrm{Hyp}(C)) \rightarrow \mathrm{Fun}(I, C)$$

in  $\mathrm{Cat}_\infty$ , and such a map is induced by the counit  $\mathrm{Hyp}(C) \rightarrow C$ . One checks that the resulting map in  $\mathrm{Cat}_\infty^{hC_2}$  is an equivalence.  $\square$

**Corollary 6.4.** *There is a natural equivalence*

$$\mathcal{H}(\mathrm{Fun}(I, \mathrm{Hyp}(C)))^\sim \simeq \mathrm{Fun}(I, C)^\sim$$

in  $\mathrm{Cat}_\infty$  for  $I \in \mathrm{Cat}_\infty^{hC_2}$  and  $C \in \mathrm{Cat}_\infty$ .

It follows

**Lemma 6.5.** *For  $C \in \mathrm{Cat}_\infty^{\mathrm{st}}$  the simplicial object  $S_\bullet^{e,\sim}(\mathrm{Hyp}(C))_h$  in  $\mathrm{Spc}$  is naturally equivalent to the simplicial object  $S_\bullet^{e,\sim}(C)$ .*

**Proposition 6.6.** *For  $C \in \mathrm{Cat}_\infty^{\mathrm{st}}$  there is a natural equivalence  $\mathrm{GW}(\mathrm{Hyp}(C)) \simeq K(C)$  of grouplike  $E_\infty$ -spaces.*

*Proof.* Note first that for any simplicial space  $X$  the natural map  $E(X) \rightarrow X$  induces an equivalence  $|E(X)| \rightarrow |X|$  (this follows from [15, Lemma 1]). Thus by Lemma 6.5  $\mathrm{GW}(\mathrm{Hyp}(C))$  is given as the fiber of the map

$$|S_\bullet^{e,\sim}(C)| \rightarrow |S_\bullet^{e,\sim}(C)| \times |S_\bullet^{w,\sim}(C^{\mathrm{op}})| \simeq |S_\bullet^{e,\sim}(C)| \times |S_\bullet^{e,\sim}(C)|$$

which is naturally equivalent to the diagonal. The claim follows.  $\square$

## 7 A connective real $K$ -theory spectrum

Recall from [10, §6] the left adjoints

$$\mathrm{Cat} \xrightarrow{f} \mathrm{Cat}^\Sigma \xrightarrow{l} \mathrm{Cat}^{\mathrm{preadd}},$$

where  $l$  is a localization. The functor  $f$  sends a small  $\infty$ -category  $C$  to the full subcategory of  $\mathcal{P}(C)$  which contains the essential image of  $C$  under the Yoneda embedding  $C \rightarrow \mathcal{P}(C)$  and is closed under finite coproducts, see the proof of [14, Proposition 5.3.6.2].

For  $C \in \mathrm{Cat}_\infty^\Sigma$  we let  $\mathcal{P}_\Sigma(C) := \mathrm{Fun}^\Pi(C^{\mathrm{op}}, \mathrm{Spc})$ , see [14, §5.5.8]. For  $C \in \mathrm{Cat}_\infty$  we let

$$\mathcal{P}^\oplus(C) := \mathrm{Fun}(C^{\mathrm{op}}, E_\infty(\mathrm{Spc})) \simeq E_\infty(\mathcal{P}(C))$$

(for the last equivalence see [13, Remark 2.1.3.4]), and for  $C \in \mathbf{Cat}_\infty^\Sigma$  we set

$$\mathcal{P}_\Sigma^\oplus(C) := \mathrm{Fun}^\Pi(C^{\mathrm{op}}, E_\infty(\mathbf{Spc})) \simeq E_\infty(\mathcal{P}_\Sigma(C)).$$

If  $C \in \mathbf{Cat}_\infty^{\mathrm{preadd}}$  then  $\mathcal{P}_\Sigma^\oplus(C) \simeq \mathcal{P}_\Sigma(C)$ , see [6, Corollary 2.5 (iii)].

For  $C \in \mathbf{Cat}_\infty$  we have a canonical left adjoint  $\mathcal{P}(C) \rightarrow \mathcal{P}^\oplus(C)$ .

For  $C \in \mathbf{Cat}_\infty^\Sigma$  we have a natural square of left adjoints

$$\begin{array}{ccc} \mathcal{P}(C) & \longrightarrow & \mathcal{P}_\Sigma(C) \\ \downarrow & & \downarrow \\ \mathcal{P}^\oplus(C) & \longrightarrow & \mathcal{P}_\Sigma^\oplus(C) \end{array}$$

which commutes up to a natural equivalence since the corresponding right adjoints do. In particular we exhibit a natural functor  $C \rightarrow \mathcal{P}_\Sigma^\oplus(C)$ .

For  $C \in \mathbf{Cat}_\infty$  let  $F^\oplus(C)$  be the smallest full subcategory of  $\mathcal{P}^\oplus(C)$  that contains the essential image of  $C \rightarrow \mathcal{P}^\oplus(C)$  and is closed under finite coproducts. Note  $F^\oplus(C)$  is preadditive. Similarly for  $C \in \mathbf{Cat}_\infty^\Sigma$  let  $F_\Sigma^\oplus(C)$  be the essential image of  $C \rightarrow \mathcal{P}_\Sigma^\oplus(C)$ .  $F_\Sigma^\oplus(C)$  is also preadditive.

**Proposition 7.1.** *i) There is a natural equivalence of functors  $l \circ f \simeq F^\oplus$ , so for  $C \in \mathbf{Cat}_\infty$  the category  $F^\oplus(C)$  is the free preadditive category on the  $\infty$ -category  $C$ .*

*ii) There is a natural equivalence of functors  $l \simeq F_\Sigma^\oplus$ , so for  $C \in \mathbf{Cat}_\infty^\Sigma$  the category  $F_\Sigma^\oplus(C)$  is the free preadditive category on the  $\infty$ -category  $C$  with finite coproducts.*

*Proof.* The first point is [7, Proposition 2.8], the second point follows similarly.  $\square$

For a small  $\infty$ -category  $C$  which has pullbacks we denote by  $\mathrm{Span}(C)$  the  $\infty$ -category of spans in  $C$ . That is the  $\infty$ -category denoted  $A^{\mathrm{eff}}(C)$  in [1, Definition 3.6].

The assignment  $C \mapsto \mathrm{Span}(C)$  can be viewed as a functor

$$\mathrm{Span}: \mathbf{Cat}_\infty^{\mathrm{lex}} \rightarrow \mathbf{Cat}_\infty,$$

where  $\mathbf{Cat}_\infty^{\mathrm{lex}}$  is the subcategory of  $\mathbf{Cat}_\infty$  of  $\infty$ -categories with all finite limits and left exact functors between them (see loc. cit.).

We denote by  $\mathbf{Fin}$  the  $\infty$ -category of finite sets.

**Proposition 7.2.** *The  $\infty$ -category  $\mathrm{Span}(\mathbf{Fin})$  is preadditive, and the natural functor  $F^\oplus(*) \rightarrow \mathrm{Span}(\mathbf{Fin})$  sending the point to the one element set is an equivalence.*

*Proof.* This follows by comparing mapping spaces.  $\square$

We recall the

**Theorem 7.3.** *Let  $G$  be a finite group. Then the symmetric monoidal  $\infty$ -category of genuine  $G$ -spectra is equivalent to the full subcategory*

$$\mathrm{Fun}^{\mathrm{II}}(\mathrm{Span}(\mathbf{Fin}[G]), \mathrm{Sp}) \subset \mathrm{Fun}(\mathrm{Span}(\mathbf{Fin}[G]), \mathrm{Sp})$$

*of finite coproduct preserving functors equipped with the symmetric monoidal structure which is induced by the Day convolution product on the functor category.*



*Proof.* The equivalence is [9], [1, Example B.6], the symmetric monoidal structure is (partially) discussed in [2, §3].  $\square$

**Lemma 7.4.** *Let  $C$  be a cocomplete  $\infty$ -category. Then the functor  $F: C \rightarrow \text{Fun}(\text{Spc}, C)$  which sends  $X \in C$  to the functor  $\text{Spc} \ni K \mapsto K \otimes C$  is left adjoint to the functor  $\text{Fun}(\text{Spc}, C) \ni \varphi \mapsto \varphi(*)$ .*

*Proof.* This follows from the fact that for any  $X \in C$  the functor  $F(X)$  is the left Kan extension of the functor  $* \rightarrow C$  which sends the unique object of  $*$  to  $X$  along the inclusion  $* \rightarrow \text{Spc}$  which sends the object of  $*$  to  $*$ .  $\square$

Let  $C \in \text{Cat}_\infty^{\text{lex}}$  and  $\varphi: \text{Spc} \rightarrow \text{Cat}_\infty^{\text{op}}$  the functor  $K \mapsto \text{Span}(\text{Fun}(K, C))$  (note that  $K \mapsto \text{Fun}(K, C)$  is a functor  $\text{Spc} \rightarrow (\text{Cat}_\infty^{\text{lex}})^{\text{op}}$ ). Then  $\text{Span}(C) \simeq \varphi(*)$ , and this equivalence is by Lemma 7.4 adjoint to a map  $F \rightarrow \varphi$  in  $\text{Fun}(\text{Spc}, \text{Cat}_\infty^{\text{op}})$ , where  $F$  is defined by  $\text{Spc} \ni K \mapsto \text{Fun}(K, \text{Span}(C))$ .

In particular this exhibits for any  $K \in \text{Spc}$  a natural functor

$$f_C^K: \text{Span}(\text{Fun}(K, C)) \rightarrow \text{Fun}(K, \text{Span}(C)).$$

For a finite group  $G$  we set  $g_G := f_{\text{Fin}}^{BG}$ , so we have

$$g_G: \text{Span}(\text{Fin}[G]) \rightarrow \text{Span}(\text{Fin})[G].$$

**Lemma 7.5.** *The  $\infty$ -category  $(\text{Cat}_\infty^{hC_2})^{\text{st}}$  is preadditive.*

So by Proposition 7.2 for  $C \in (\text{Cat}_\infty^{hC_2})^{\text{st}}$  we exhibit a natural functor

$$\text{Span}(\text{Fin}) \rightarrow (\text{Cat}_\infty^{hC_2})^{\text{st}}$$

sending the generator to  $C$ , which induces the second map in the composition

$$\text{Span}(\text{Fin}[C_2]) \xrightarrow{g_{C_2}} \text{Span}(\text{Fin})[C_2] \rightarrow (\text{Cat}_\infty^{hC_2})^{\text{st}}[C_2] \rightarrow (\text{Cat}_\infty^{hC_2})^{\text{st}} \xrightarrow{\text{GW}} \text{Sp},$$

whereas the third map is the map  $t_{\text{Cat}_\infty^{\text{st}}}$  described in [10, §9] (and GW takes values in grouplike  $E_\infty$ -spaces aka connective spectra).

By Theorem 7.3 we obtain a genuine  $C_2$ -spectrum which we denote by  $\text{KR}(C)$  and call the connective real  $K$ -theory spectrum of  $C$ .

**Proposition 7.6.** *For  $C \in (\text{Cat}_\infty^{hC_2})^{\text{st}}$  the underlying spectrum of  $\text{KR}(C)$  is naturally equivalent to the  $K$ -theory spectrum  $K(C)$  (which thus inherits a natural  $C_2$ -action), and the  $C_2$ -fixed points of  $\text{KR}(C)$  are canonically equivalent to  $\text{GW}(C)$ .*

*Proof.* In the defining composition of  $\text{KR}(C)$  the  $C_2$ -set  $*$  is sent to the  $C_2$ -fixed points. But the image of  $*$  in  $(\text{Cat}_\infty^{hC_2})^{\text{st}}[C_2]$  is  $C$  with the trivial  $C_2$ -action yielding the second claim.

The underlying spectrum of  $\text{KR}(C)$  is the image of the  $C_2$ -orbit  $C_2$ . Its image in  $(\text{Cat}_\infty^{hC_2})^{\text{st}}[C_2]$  is  $C \times C$  with the  $C_2$ -action which switches the two factors. The resulting object  $t_{\text{Cat}_\infty^{\text{st}}}(C \times C)$  of  $(\text{Cat}_\infty^{hC_2})^{\text{st}}$  is then seen to be naturally equivalent to  $\text{Hyp}(C)$ , so the first claim follows from Proposition 6.6.  $\square$

## References

- [1] Clark Barwick. Spectral Mackey functors and equivariant algebraic K-theory (I). arXiv:1404.0108, to appear in *Advances in mathematics*.
- [2] Clark Barwick. Spectral Mackey functors and equivariant algebraic K-theory (II). arXiv:1505.03098.
- [3] Daniel Dugger. Classification spaces of maps in model categories. Preprint, <http://pages.uoregon.edu/ddugger/>.
- [4] W. G. Dwyer and D. M. Kan. Calculating simplicial localizations. *J. Pure Appl. Algebra*, 18(1):17–35, 1980.
- [5] W. G. Dwyer and D. M. Kan. Function complexes in homotopical algebra. *Topology*, 19(4):427–440, 1980.
- [6] David Gepner, Moritz Groth, and Thomas Nikolaus. Universality of multiplicative infinite loop space machines. *Algebr. Geom. Topol.*, 15(6):3107–3153, 2015.
- [7] Saul Glasman. Goodwillie calculus and Mackey functors. arXiv:1610.03127.
- [8] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*. Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition [MR1711612].
- [9] Bertrand Guillou and Peter May. Models of G-spectra as presheaves of spectra. arXiv:1110.3571v3.
- [10] Hadrian Heine, Alejo Lopez-Avila, and Markus Spitzweck. Infinity categories with duality and hermitian multiplicative infinite loop space machines. Preprint.
- [11] Jacob Lurie. Course notes for Math 287x, Lecture 4. available at <http://www.math.harvard.edu/~lurie/>.
- [12] Jacob Lurie. Course notes for Math 287x, Lecture 5. available at <http://www.math.harvard.edu/~lurie/>.
- [13] Jacob Lurie. Higher Algebra. available at <http://www.math.harvard.edu/~lurie/>.
- [14] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.
- [15] Marco Schlichting. The Mayer-Vietoris principle for Grothendieck-Witt groups of schemes. *Invent. Math.*, 179(2):349–433, 2010.
- [16] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

Fakultät für Mathematik, Universität Osnabrück, Germany.

e-mail:

[markus.spitzweck@uni-osnabrueck.de](mailto:markus.spitzweck@uni-osnabrueck.de)